On the Metric Structure of Space-Time*

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Abstract

I present an analysis of the physical assumptions needed to obtain the metric structure of space-time. For this purpose I combine the axiomatic approach pioneered by Robb with ideas drawn from works on Weyl's *Raumproblem*. The concept of a Lorentzian manifold is replaced by the weaker concept of an 'event manifold', defined in terms of volume element, causal structure and affine connection(s). Exploiting properties of its structure group, I show that distinguishing Lorentzian manifolds from other classes of event manifolds requires the key idea of General Relativity: namely that the manifold's physical structure, rather than being fixed, is itself a variable.

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1 Introduction

In General Relativity, space-time is assumed to be a Lorentzian manifold. The metric field determines the volume element and the causal structure. Conversely, given a Lorentzian manifold, the causal structure and the volume element uniquely specify the metric field. But why does one start from a Lorentzian manifold in the first place? Why are light signals described by a quadratic equation $g_{ab}dx^a dx^b = 0$? Why is the light 'cone' not a pyramid standing on its top? One might argue that the quadratic term is the lowest order contribution to the Taylor expansion of some distance function; but it is unclear both how this distance function is motivated and, if it is, whether the quadratic term is necessarily non-trivial. Indeed there have been suggestions that rather than being Lorentzian, the space-time geometry might be Finslerian [1]. So why do Lorentzian manifolds nevertheless play a privileged role?

This is a particular case of Weyl's space problem. It was first solved by Weyl [2], who also placed it in the context of General Relativity. A more elegant treatment was given by Cartan [3]. Since then their result, known as Weyl-Cartan theorem, has been reviewed by various authors [4]. A very different line of reasoning was initiated by Robb, who for the case of Special Relativity derived the Minkowskian metric from properties of the causal relations *before* and *after* [5]. More recent attempts to axiomatize General Relativity in Robb's spirit are based on such notions as signals, light rays, freely falling particles, or clocks [6]; some even invoke quantum mechanics [7].

A synthesis of these two kinds of approaches is the aim of the present paper. Its purpose is not so much an actual derivation as it is an analysis: which physical assumptions are being tacitly made whenever one postulates the existence of a Lorentzian metric? Only after these assumptions are exhibited can one start to systematically relax them; thus, answers to the above question may be helpful for the study of more general space-time structures.

Primitive concepts are taken to be events, counting of events, causal relationships and the ability to compare measurements; the corresponding mathematical structures are a differentiable manifold, volume element, causal vectors and affine connection(s), leading to the notion of an 'event manifold'. The key assumption, which I will call 'deformability', is that the event manifold's physical structure is allowed to vary freely. The proof of the Weyl-Cartan theorem is then reviewed to establish the result that any deformable

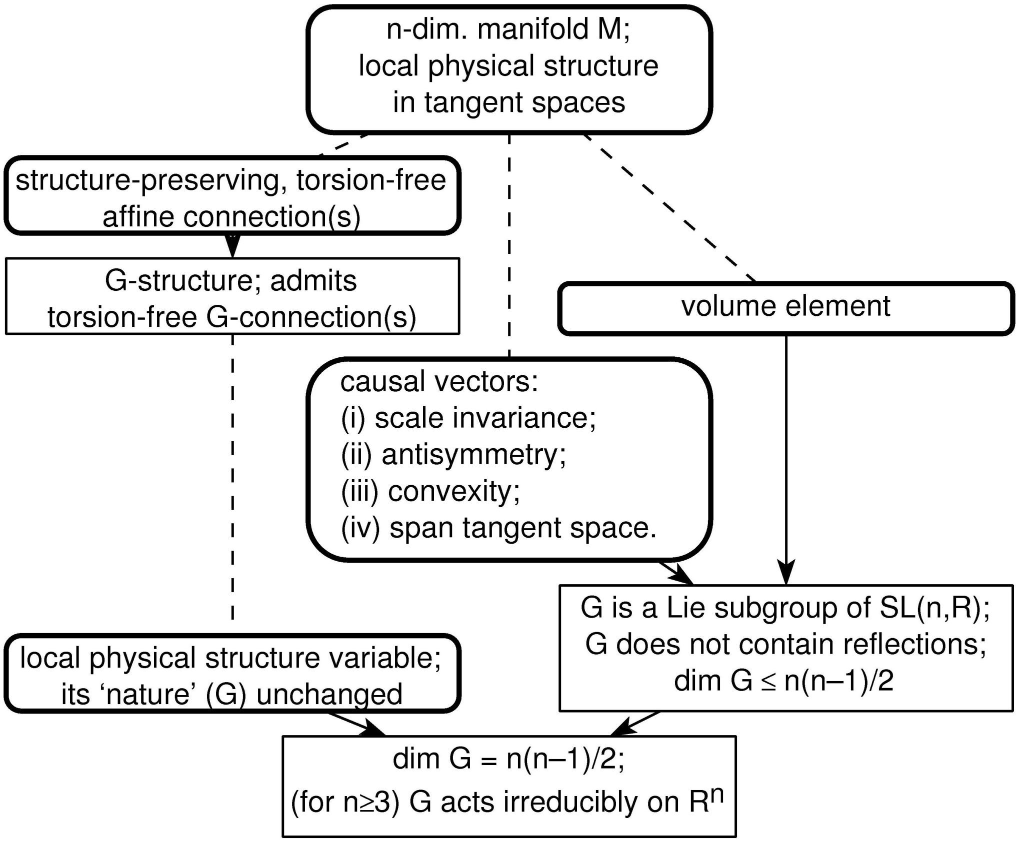

Figure 1: Line of argument.

event manifold must be Lorentzian.

2 Event Manifolds

I assume that space-time is a connected n-dimensional differentiable manifold M. At $x \in M$, local measurements (e.g., evaluating vector fields) are performed using a basis of the tangent space T_xM. In order to have a means to compare local measurements at different points, I require the manifold to be endowed with an affine connection. The connection is assumed to be torsion-free. Space-time is also endowed with a physical structure Φ reflecting, e.g., causal relationships between events. Details of this structure do not

matter at this point; all I assume is that (a) Φ induces in each T_xM a local physical structure $T_x\Phi$, and (b) this local structure can be measured with a basis of T_xM. In order for an observer at $x \in M$ to be able to determine the local structure not only at x itself but also elsewhere on the manifold, the connection must preserve the local physical structure.

A frame field provides a map from the local physical structure in T_xM to some structure in R^n for every $x \in M$. The images in R^n will generally vary from point to point. However, since the connection is structure-preserving, one can use parallel transport to construct a special frame field called an n-bein such that the image of the local physical structure is the same everywhere. This enables one to fix a standard structure η in R^n and then describe the local physical structure by the n-bein. However, there may be several n-beins associated with the same local physical structure. Such a redundancy in the mathematical description of a physical structure gives rise to a gauge theory. Its symmetry group is the Lie subgroup G of $GL(n, R)$ which leaves η invariant. Hence, the local physical structure on the manifold M corresponds to an entire subbundle of the frame bundle $F(M)$, with (reduced) structure group G. This is a G-structure on M [8]. It must admit a connection, called a G-connection. In contrast to Weyl, I do not assume the G-connection to be determined uniquely.

A discrete rather than continuous set of events can be characterized by the number of events and their mutual causal relationships. By analogy one expects that on a continuous space-time manifold, the physical structure Φ should be a pair $(\mu, \leq)$ consisting of a volume element and a causal relation. A differentiable map $\gamma : [0, 1] \to M$ with the property that $\gamma(t_1) \leq \gamma(t_2)$ iff $t_1 \leq t_2$ is called a causal curve. Φ then induces in each tangent space T_xM a local physical structure $T_x\Phi$, consisting of (a) a volume element and (b) the set of vectors, called causal vectors, which are tangent to a causal curve through x. Correspondingly, the standard structure η in R^n consists of (a) a volume element and (b) the image j^+ of the set of causal vectors.

About the sets j^+ and $j^- := -j^+$ I make the following four assumptions. (i) Except for its direction, the parametrization of a causal curve is irrelevant; hence j^+ is scale invariant. (ii) But the well-defined direction of causal curves distinguishes locally between past and future; therefore $j^+ \cap j^- = \{0\}$ ('antisymmetry'). (iii) As there is no torsion, two causal vectors at $x \in M$ can be used to construct an infinitesimal closed geodesic parallelogram; and as the connection is structure-preserving, all its four sides must be parts of causal curves. Requiring transitivity of the causal relation, its (geodesic)

diagonal, too, is part of a causal curve; thus j^+ is convex. (iv) Let θ be the coframe field dual to the n-bein. 'No torsion' implies that for any vector fields X, Y,

$$\nabla_X \theta(Y) - \nabla_Y \theta(X) = \theta([X,Y]) \quad . \tag{1}$$

Defining $S := \mathrm{span}\{j^+\}$, the requirement that the connection be structure-preserving implies $\nabla_U S \subseteq S$ for any vector field U. Choosing $X, Y \in \theta^{-1}(S)$ in (1) one thus obtains

$$[\theta^{-1}(S), \theta^{-1}(S)] \subseteq \theta^{-1}(S) \quad . \tag{2}$$

By Frobenius' theorem, the causal curves mesh to form a foliation of M, each leaf having dimension at most $\dim S$. Different leaves would represent 'separated worlds', a situation I want to exclude. Hence $\dim S = n$, and j^+ spans the entire R^n.

The above properties of the local physical structure are sufficient to determine the structure group G for $n = 2$. In this case, the boundary ∂j^+ consists of two straight rays. Choosing basis vectors on these rays, any symmetry transformation Λ has the form $\Lambda = \mathrm{diag}\,(\lambda, 1/\lambda)$ with $\lambda > 0$. The group of these transformations is isomorphic to $SO(1,1)$. Although for $n \geq 3$ the structure group G is not yet determined, one may already infer some properties: as a structure group, G is a Lie group; it preserves the volume and is therefore a subgroup of $SL(n,R)$; it preserves j^+ and thus may not contain reflections: $(-1) \notin G$; finally,

$$\dim G \leq n(n-1)/2 \quad . \tag{3}$$

Proof. I prove by induction that any invertible linear map $g : R^n \to R^n$ preserving η is determined by at most $n(n-1)/2$ parameters. (i) The proposition holds for $n = 2$. (ii) Assume it is proven for n. In $n+1$ dimensions, g is specified by the images of $n+1$ linearly independent vectors $v_1, \ldots, v_{n+1}$. These vectors can be chosen such that v_{n+1} is the unique intersection $v_{n+1} = \partial j^+ \cap (v_1 + \partial j^-) \cap \ldots \cap (v_n + \partial j^-)$. Since g preserves $\partial j^\pm$, the image of v_{n+1} is uniquely determined by the images of $v_1, \ldots, v_n$. It is therefore sufficient to consider the restriction of g to $S := \mathrm{span}\{v_1, \ldots, v_n\}$. And since (by a suitable choice of the v_i) $\partial j^+_s := S \cap \partial j^+$ can be made to have all the properties of a 'light cone' in S, it is sufficient to determine the images only of vectors that lie on ∂j^+_s. To specify $g(\partial j^+_s) = \partial j^+ \cap g(S)$ requires at most n parameters; to specify how vectors transform within ∂j^+_s requires, by assumption, at most $n(n-1)/2$; so altogether at most $(n+1)n/2$. Q.E.D.

3 Deformability

So far my considerations have been very general, and the symmetry group G is by no means uniquely determined. Only now the key idea of General Relativity comes into play: rather than being fixed as in Newtonian theory, the local physical structure on the space-time manifold is itself a *variable*; it depends on the distribution of matter in the universe (and on boundary conditions). Whenever the local physical structure is thus allowed to vary freely I call the event manifold 'deformable'. However, the 'nature' of the physical structure — embodied by η — and hence the symmetry group G must remain unchanged. Provided the local physical structure reflects the distribution of matter, deformability amounts to the requirement that arbitrary matter distributions be allowed.

Mathematically, varying the G-structure corresponds to varying the n-bein; or, equivalently, its dual θ. By assumption, any choice of the G-structure, and hence of θ, must admit a torsion-free G-connection. If expressed with respect to θ, the connection 1-form ω takes values in Lie(G) and thus has $n \cdot \dim G$ degrees of freedom. The requirement that it be torsion-free reads

$$d\theta + \omega \wedge \theta = 0 \quad . \tag{4}$$

Since $\omega \wedge \theta$ is an R^n-valued 2-form, this requirement imposes $n \cdot n(n-1)/2$ constraints on ω which for arbitrary θ can only be satisfied if $n(n-1)/2 \leq \dim G$. Together with (3) one thus obtains

$$\dim G = n(n-1)/2 \quad . \tag{5}$$

This result implies that the connection on M is indeed unique, which Weyl had assumed without proof.

Let us assume that G leaves a proper subspace $S \subset R^n$ invariant. This again leads to 2, which for $\dim S \geq 2$ is an undue restriction on θ. Now suppose S is spanned by a vector $r \in R^n$. The invariance of S implies that there is a 1-form α such that $\omega r = r \otimes \alpha$. Defining $X := \theta^{-1}(r)$ and using (4) this yields

$$L_X\theta + \omega(X)\theta = r \otimes \alpha \quad . \tag{6}$$

Here L denotes the Lie derivative. Defining the maps

$$\Lambda : R^n \to R^n \quad , \quad \Lambda u = -(L_X\theta)(\theta^{-1}(u)) \tag{7}$$

$$\xi : R^n \to R \quad , \quad \xi(u) = \alpha(\theta^{-1}(u)) \tag{8}$$

one obtains

$$\omega(X) - r \otimes \xi = \Lambda \quad , \tag{9}$$

a relation among endomorphisms of R^n. Since θ may be varied freely, Λ can be chosen freely with the sole constraint that $\Lambda r = 0$. Thus, restricting (9) to an $(n-1)$-dimensional space $\overline{S}$ complement to S yields an equation which in general has a solution only if $\omega(X)|_{\overline{S}}$ and $\xi|_{\overline{S}}$ together have at least as many degrees of freedom as $\Lambda|_{\overline{S}}$. Hence $\dim G + (n-1) \geq n(n-1)$ and therefore $\dim G \geq (n-1)^2$, a condition which is compatible with $\dim G = n(n-1)/2$ only if $n = 2$. Thus for $n \geq 3$, G acts irreducibly on R^n.

I have now established several important properties of the symmetry group G which are summarized in figure 1. It can be shown that these properties uniquely determine the Lorentz group, which in turn implies the existence of an invariant metric with signature $n-2$.

4 Conclusion

Any deformable event manifold is Lorentzian.

This result has a nice physical interpretation if one assumes a one-to-one correspondence between the manifold's local physical structure and the distribution of matter: out of all possible event manifolds, only Lorentzian manifolds admit arbitrary matter distributions; any non-metric structure imposes undue restrictions.

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